

P-V criticality of conformal anomaly corrected AdS black holes

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Abstract. The effects of conformal anomaly on the thermodynamics of black holes are investigated in this Letter from the perspective of $P - V$ criticality of AdS black holes. Treating the cosmological constant as thermodynamic pressure, we extend the recent research to the extended phase space. Firstly, we study the $P-V$ criticality of the uncharged AdS black holes with conformal anomaly and find that conformal anomaly does not influence whether there exists Van der Waals like critical behavior. Secondly, we investigate the $P-V$ criticality of the charged cases and find that conformal anomaly influences not only the critical physical quantities but also the ratio $\frac{P_c r_c}{T_c}$. The ratio is no longer a constant as before but a function of conformal anomaly parameter $\tilde{\alpha}$. We also show that the conformal parameter should satisfy a certain range to guarantee the existence of critical point that has physical meaning. Our results show the effects of conformal anomaly.

1. Introduction

Conformal anomaly is such an important concept that it has lots of applications in quantum field theory, cosmology and black hole physics. So it would be of great interest to investigate its effects on the thermodynamics of black holes. Many efforts have been devoted to this issue. Cai et al. [1] investigated the thermodynamics of static and spherically symmetric black holes with conformal anomaly. In addition to Bekenstein-Hawking area entropy, an extra logarithmic correction was found. The entropy was also studied by Li who took an approach of quantum tunneling [2]. Considering the back reaction through the conformal anomaly, phase transitions of Schwarzschild black hole were investigated [3] and an additional phase transition was discovered. Ehrenfest equations were also studied [4]. Moreover, we studied the phase structures of black holes with conformal anomaly [5] and found that they have much richer phase structures than black holes without conformal anomaly. Recently, Cai [6] obtained analytical AdS black hole solutions with conformal anomaly. Hawking-Page transition was investigated and novel features due to conformal anomaly were found in black holes with a negative

constant curvature horizon. Although many efforts have been made, all of them were carried out in the non-extended phase space. In this Letter, we would like to extend the research of Ref. [6] to the extended phase space, hoping to observe more evidences of the effects of conformal anomaly on the thermodynamics of black holes.

The extended phase space refers to the phase space which includes the pressure and volume as thermodynamic variables. Thermodynamics in the extended phase space has gained more and more attention recently [7]- [39]. The main idea is to treat the cosmological constant as thermodynamic pressure and the conjugate quantity as volume. The motivation comes from the consideration of variation of cosmological constant to make the first law of black hole thermodynamics consistent with the Smarr relation. The critical behavior of black holes in the extended phase space, especially $P - V$ criticality, has been extensively investigated [13]- [37]. For nice review, see Ref. [28, 39].

The outline of this Letter is as follows. In Sec.2, thermodynamics of AdS black holes with conformal anomaly will be briefly studied in the extended phase space. Then P - V criticality of the uncharged AdS black holes with conformal anomaly will be investigated in Sec.3 while the charged cases will be discussed in Sec.4. To observe the effects of conformal anomaly, comparisons will be made between AdS black holes with conformal anomaly and those without it. In the end, conclusions will be drawn in Sec.5.

2. Thermodynamics of conformal anomaly corrected AdS black hole in the extended phase space

The newly derived conformal anomaly corrected AdS black holes' metric can be written as [6]

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_{2k}^2, \quad (1)$$

where

$$f(r) = k - \frac{r^2}{4\tilde{\alpha}} \left(1 - \sqrt{1 + \frac{8\tilde{\alpha}}{l^2} - \frac{16\tilde{\alpha}GM}{r^3} + \frac{8\tilde{\alpha}Q^2}{r^4}} \right). \quad (2)$$

$d\Omega_{2k}^2$ denotes the line element of two dimensional Einstein space with constant scalar curvature $2k$, where k can be taken as 0, 1 or -1 without loss of generality [6]. It was argued that the integration constant M and Q should be interpreted as the mass of black holes and the U(1) conserved charge of the conformal field theory respectively [6]. When $\tilde{\alpha} \rightarrow 0$ [6], the solutions reduces to the Reissner-Nordström-AdS (RN-AdS) black holes as

$$f(r) = k + \frac{r^2}{l^2} - \frac{2GM}{r} + \frac{Q^2}{r^2}. \quad (3)$$

One can obtain the event horizon radius r_+ by solving the equation $f(r_+) = 0$ for the largest root. With the event horizon radius, one can derive the expression of the mass of black holes as

$$M = \frac{r_+^4 + kl^2r_+^2 - 2k^2l^2\tilde{\alpha} + l^2Q^2}{2l^2r_+}. \quad (4)$$

Note that G has been set to one. The Hawking temperature can be obtained as

$$T = \frac{f'(r_+)}{4\pi} = \frac{r_+}{4\pi(r_+^2 - 4k\tilde{\alpha})} \left(k + \frac{3r_+^2}{l^2} - \frac{Q^2}{r_+^2} + \frac{2k^2\tilde{\alpha}}{r_+^2} \right). \quad (5)$$

The entropy can be derived as

$$S = \int \frac{1}{T} \left(\frac{\partial M}{\partial r_+} \right) dr_+ = \pi r_+^2 - 8\pi k\tilde{\alpha} \ln r_+ + S_0. \quad (6)$$

Thermodynamic pressure and volume in the extended phase space can be defined as

$$P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi l^2}, \quad (7)$$

$$V = \left(\frac{\partial M}{\partial P} \right)_{S,Q}. \quad (8)$$

Note that the second equation in Eq.(7) holds for four dimensional black holes. Utilizing Eqs. (4) and (7), one can reorganize the expression of the mass as

$$M = \frac{8P\pi r_+^4 + 3kr_+^2 - 6k^2\tilde{\alpha} + 3Q^2}{6r_+}, \quad (9)$$

from which one can easily derive the thermodynamic volume as

$$V = \frac{4\pi r_+^3}{3}. \quad (10)$$

Note that the entropy gains an extra logarithmic term due to the effect of conformal anomaly while the thermodynamic volume is the same as that of RN-AdS black holes. It is quite easy to verify that the first law of black hole thermodynamics in the extended phase space and the Smarr relation can be written as

$$dM = TdS + \Phi dQ + VdP, \quad (11)$$

$$M = 2TS + \Phi Q - 2VP, \quad (12)$$

where

$$\Phi = \left(\frac{\partial M}{\partial Q} \right)_{S,P} = \frac{Q}{r_+}. \quad (13)$$

3. P - V criticality of uncharged AdS black holes with conformal anomaly

In this section, we focus on the uncharged case $Q = 0$. Substituting Eq. (7) and $Q = 0$ into Eq. (5), we obtain the Hawking temperature for the uncharge case as

$$T = \frac{kr_+^2 + 8P\pi r_+^4 + 2k^2\tilde{\alpha}}{4\pi r_+(r_+^2 - 4k\tilde{\alpha})}, \quad (14)$$

from which we can derive the equation of state as

$$P = \frac{T}{2r_+} - \frac{k}{8\pi r_+^2} - \frac{2kT\tilde{\alpha}}{r_+^3} - \frac{2k^2\tilde{\alpha}}{8\pi r_+^4}. \quad (15)$$

The critical point can be defined as follows

$$\left. \frac{\partial P}{\partial r_+} \right|_{r=r_c, T=T_c} = 0, \quad (16)$$

$$\left. \frac{\partial^2 P}{\partial r_+^2} \right|_{r=r_c, T=T_c} = 0, \quad (17)$$

where the subscript "c" denotes the physical quantities at the critical point. Utilizing Eqs. (15) and (16), one can get

$$T_c = \frac{kr_c^2 + 4k^2\tilde{\alpha}}{2\pi r_c(r_c^2 - 12k\tilde{\alpha})}, \quad (18)$$

Solving Eq. (17) and then substituting Eq. (18) into the result, one can obtain

$$k(r_c^4 + 24k\tilde{\alpha}r_c^2 - 48k^2\tilde{\alpha}^2) = 0. \quad (19)$$

Except $k = 0$ (we would not consider this case for the Hawking temperature is zero), Eq. (19) has two positive roots for r_c , namely,

$$r_c = 2\sqrt{(-3k \mp 2\sqrt{3})k\tilde{\alpha}}, \quad (20)$$

where "−" corresponds to the case $k = -1$ while "+" corresponds to the case $k = 1$. However, substituting Eq. (20) back into Eq. (18), one can derive that

$$T_c = -\frac{\sqrt{(3 \mp 2\sqrt{3})k\tilde{\alpha}}}{12\pi\tilde{\alpha}}, \quad (21)$$

where "−" corresponds to the case $k = -1$ while "+" corresponds to the case $k = 1$. It is obvious that the Hawking temperature at these two critical points is negative. So these two critical points do not make any sense physically and there would be no Van der Waals like critical behavior for the uncharged case. This is in accord with Schwarzschild AdS black holes, implying that the conformal anomaly does not influence whether there exists Van der Waals like critical behavior.

4. P - V criticality of charged AdS black holes with conformal anomaly

Utilizing Eqs. (5) and (7), one can obtain the equation of state of charged AdS black holes with conformal anomaly as

$$P = \frac{T}{2r_+} - \frac{k}{8\pi r_+^2} - \frac{2kT\tilde{\alpha}}{r_+^3} + \frac{Q^2 - 2k^2\tilde{\alpha}}{8\pi r_+^4}. \quad (22)$$

Substituting Eq. (22) into Eq. (16), one can get

$$T_c = \frac{kr_c^2 + 4k^2\tilde{\alpha} - 2Q^2}{2\pi r_c(r_c^2 - 12k\tilde{\alpha})}. \quad (23)$$

Utilizing Eqs. (17), (22) and (23), one can obtain

$$kr_c^4 + (24k^2\tilde{\alpha} - 6Q^2)r_c^2 - 48k^3\tilde{\alpha}^2 + 24kQ^2\tilde{\alpha} = 0. \quad (24)$$

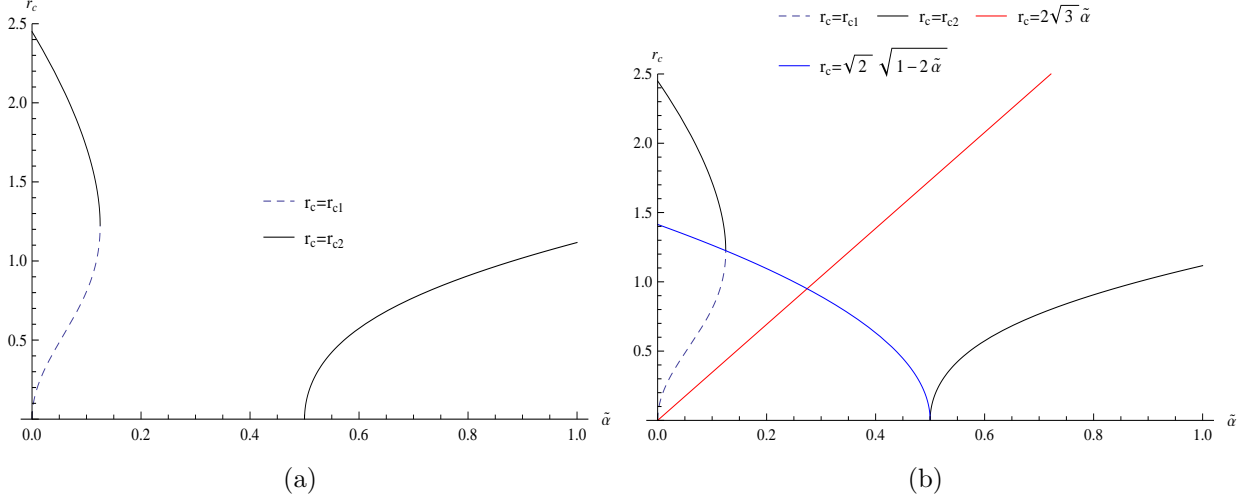


Figure 1. (a) r_c vs. $\tilde{\alpha}$ for $k = 1, Q = 1$ (b) Curve constrained by positive Hawking temperature

Solving the above equation, one can obtain the positive roots as

$$r_c = \sqrt{\frac{3Q^2}{k} - 12k\tilde{\alpha} \mp \frac{\sqrt{3}\sqrt{3Q^4 - 32k^2Q^2\tilde{\alpha} + 64k^4\tilde{\alpha}^2}}{k}}, \quad (25)$$

where “−” corresponds to r_{c1} while “+” corresponds to r_{c2} . Note that we also omit the case $k = 0$ when we solve Eq. (24) because the Hawking temperature is negative.

When $k = 1, \tilde{\alpha} = 0$, Eq. (25) reduces to $r_c = \sqrt{6}Q$, recovering the result of RN-AdS black holes in former literature [13]. When $\tilde{\alpha} \neq 0$, one can easily draw the conclusion from Eq. (25) that the location of critical point relies on the conformal anomaly parameter $\tilde{\alpha}$. To observe the effect of conformal anomaly, we fix $k = 1, Q = 1$ and show the behavior of r_c for different $\tilde{\alpha}$ in Fig.1(a). The black holes with conformal anomaly have much richer phase structure. They may have none, one or two critical points due to different $\tilde{\alpha}$. However, one should check the Hawking temperature to make sure whether the critical points are in the physical region. The positive Hawking temperature means

$$T_c|_{k=1, Q=1} = \frac{r_c^2 + 4\tilde{\alpha} - 2}{2\pi r_c(r_c^2 - 12\tilde{\alpha})} > 0, \quad (26)$$

which can be solved as

$$r_c > A \text{ or } 0 < r_c < B, \quad (27)$$

where $A = \max\{2\sqrt{3}\tilde{\alpha}, \sqrt{2-4\tilde{\alpha}}\}, B = \min\{2\sqrt{3}\tilde{\alpha}, \sqrt{2-4\tilde{\alpha}}\}$. To witness the constraint of the Hawking temperature intuitively, we plot Fig.1(b). From Eq. (27) and Fig.1(b), we find that only one critical point exists under the constraining condition of Hawking temperature. And the conformal anomaly parameter $\tilde{\alpha}$ should satisfy $0 < \tilde{\alpha} < \frac{1}{8}$.

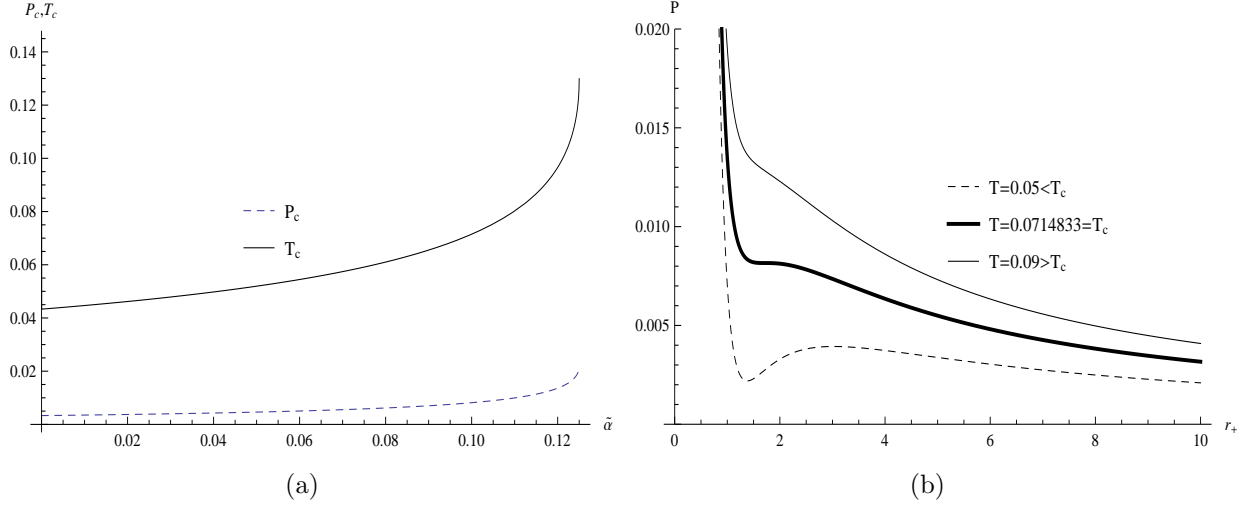


Figure 2. (a) P_c, T_c vs. $\tilde{\alpha}$ for $k=1, Q=1$ (b) P vs. r_+ for $k=1, Q=1, \tilde{\alpha}=0.1$

From Eqs. (22), (25), (26) and Fig.1(b), one can derive the explicit expressions for all the critical quantities as

$$\begin{aligned}
 r_c|_{k=1, Q=1} &= \sqrt{3 - 12\tilde{\alpha} + \sqrt{9 - 96\tilde{\alpha} + 192\tilde{\alpha}^2}}, \\
 T_c|_{k=1, Q=1} &= \frac{\sqrt{3 - 12\tilde{\alpha} + \sqrt{9 - 96\tilde{\alpha} + 192\tilde{\alpha}^2}}}{192\pi\tilde{\alpha}^2} \\
 &\quad \times (3 - 16\tilde{\alpha} - \sqrt{9 - 96\tilde{\alpha} + 192\tilde{\alpha}^2}), \\
 P_c|_{k=1, Q=1} &= \frac{6 - 18\tilde{\alpha} + \sqrt{9 - 96\tilde{\alpha} + 192\tilde{\alpha}^2}}{24\pi(3 - 12\tilde{\alpha} + \sqrt{9 - 96\tilde{\alpha} + 192\tilde{\alpha}^2})^2}.
 \end{aligned} \tag{28}$$

The behavior of P_c and T_c is depicted in Fig.2(a). From Figs.1(b) and 2(a), we can see clearly that with the increasing of $\tilde{\alpha}$, both T_c and P_c increase while r_c decreases. $P-r_+$ diagram for a specific case $\tilde{\alpha}=0.1$ is shown in Fig.2(b). When the temperature is lower than the critical temperature, the isotherm not only has stable small radius branch and stable large radius branch but also has the unstable medium radius branch. However, when the temperature is higher than the critical temperature, the above phenomenon disappears.

We also plot $P-T$ curve in Fig.3 for three different choices of $\tilde{\alpha}$. Namely, $\tilde{\alpha}=0$, $\tilde{\alpha}=0.05$, $\tilde{\alpha}=0.1$. Similar curves as the $P-T$ curve of liquid-gas phase transition can be observed. When $\tilde{\alpha}=0$, the curve recovers that of RN-AdS black hole shown in former literature [13]. The effect of conformal anomaly is reflected in different endpoints of the curves. These endpoints correspond to the critical points. Since we have demonstrate analytically that physical quantities vary with the conformal anomaly parameter, the graphical results match the analytical results quite well.

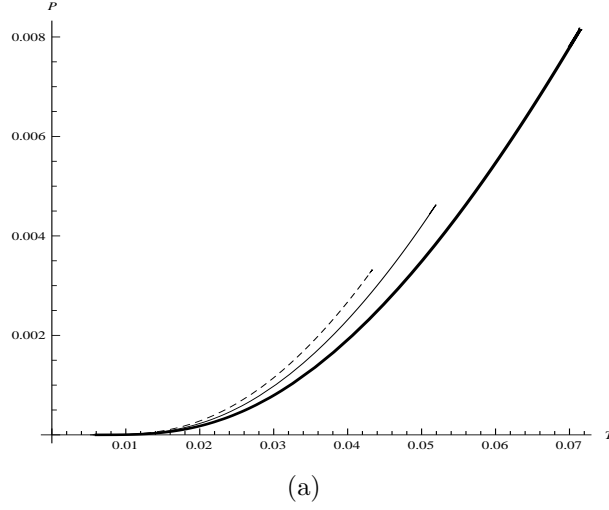


Figure 3. P vs. T when $k = 1, Q = 1$. Dashed curve for $\tilde{\alpha} = 0$, common curve for $\tilde{\alpha} = 0.05$, Thick curve for $\tilde{\alpha} = 0.1$

Now the ratio $\frac{P_c r_c}{T_c}$ can be calculated as

$$\left. \frac{P_c r_c}{T_c} \right|_{k=1, Q=1} = \frac{21 - 48\tilde{\alpha} - \sqrt{9 - 96\tilde{\alpha} + 192\tilde{\alpha}^2}}{96(1 - 2\tilde{\alpha})}. \quad (29)$$

When $\tilde{\alpha} = 0$, Eq. (29) reduces to $\frac{3}{16}$, recovering the result of RN-AdS black holes [13]. When $\tilde{\alpha} \neq 0$, the ratio is no longer a constant but a function of $\tilde{\alpha}$, showing the effect of conformal anomaly.

The case $k = -1$ can be discussed similarly. The positive Hawking temperature means

$$T_c|_{k=-1, Q=1} = \frac{-r_c^2 + 4\tilde{\alpha} - 2}{2\pi r_c(r_c^2 + 12\tilde{\alpha})} > 0, \quad (30)$$

which can be solved as

$$0 < r_c < \sqrt{4\tilde{\alpha} - 2}. \quad (31)$$

And the curve of r_c vs. $\tilde{\alpha}$ is depicted in Fig.4, which shows that no critical points exists under the constraint of the positive Hawking temperature.

5. Conclusions

To observe the effects of conformal anomaly on the thermodynamics of black holes, we focus on the $P - V$ criticality of AdS black holes with conformal anomaly. Specifically, we probe whether conformal anomaly influences the existence of Van der Waals like critical behavior and the critical physical quantities. Treating the cosmological constant as thermodynamic pressure, we extend the former research to the extended phase space. The entropy gains an extra logarithmic term due to the effect of conformal anomaly while the thermodynamic volume is the same as that of RN-AdS black holes.

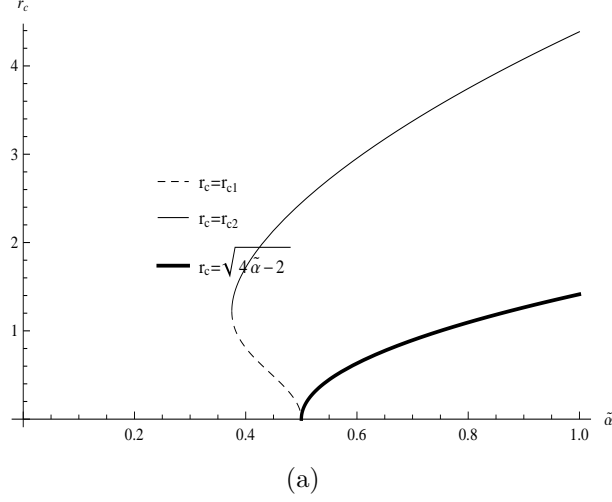


Figure 4. r_c vs. $\tilde{\alpha}$ for $k = -1, Q = 1$

Firstly, we study the P - V criticality of the uncharged AdS black holes with conformal anomaly. There are two positive roots of r_c . However, the Hawking temperature at these two critical points is negative. So these two critical points do not make any sense physically and there would be no Van der Waals like critical behavior for the uncharged case. This is in accord with Schwarzschild AdS black holes, implying that the conformal anomaly does not influence whether there exists Van der Waals like critical behavior. Secondly, we investigate the P - V criticality of the charged cases and find that conformal anomaly influences not only the critical quantities but also the ratio $\frac{P_c r_c}{T_c}$. When $k = 1$, with the increasing of conformal anomaly parameter $\tilde{\alpha}$, both T_c and P_c increase while r_c decreases. The ratio $\frac{P_c r_c}{T_c}$ is no longer a constant as before but a function of $\tilde{\alpha}$. The above results show the effects of conformal anomaly. When $k = -1$, there exists no critical points that make any sense physically.

In the context of AdS black holes, P - V criticality suggests the existence of first order small/large black hole phase transition below the critical temperature and second order phase transition at the critical point. It resembles the liquid/gas phase transition. By studying the case of nonzero alpha, we show more characteristics other than the common characteristics share by AdS black holes. Not only the critical physical quantities but also the ratio $\frac{P_c r_c}{T_c}$ are influenced due to the effect of conformal anomaly, suggesting that conformal anomaly may affect the small/large black hole phase transition. On the other hand, the curves show that the critical volume/radius exists only for some range of alpha. In this Letter, we have derived the explicit expression of this range. When the conformal anomaly parameter is above this range, one can not find root of critical point that has physical meaning. The physics is not difficult to explain. When $\tilde{\alpha} \rightarrow 0$, the black hole solutions reduces to the RN-AdS black hole. So when the conformal anomaly parameter is small enough, one may expect its critical behavior is not affected too much and similar to RN-AdS black hole. However, when the conformal anomaly parameter is above certain range, the effect of conformal anomaly begins to play an important role,

leading to different behavior from RN-AdS black hole.

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- [1] R. Cai, L. Cao, N. Ohta, *Black Holes in Gravity with Conformal Anomaly and Logarithmic Term in Black Hole Entropy*, *JHEP* **1004** (2010) 082 [arXiv:0911.4379].
- [2] R. Li, *Logarithmic entropy of black hole in gravity with conformal anomaly from quantum tunneling approach*, *Europhys.Lett.* **96** (2011) 60014 [arXiv:1112.3410].
- [3] E. J. Son, W. Kim, *Two critical phenomena in the exactly soluble quantized Schwarzschild black hole*, *JHEP* **03** (2013) 060 [arXiv:1212.2307].
- [4] J. Man, H. Cheng, *The description of phase transition in a black hole with conformal anomaly in the Ehrenfest’s scheme*, arXiv:1304.5685
- [5] J. X. Mo and W. B. Liu, Phase transitions, geometrothermodynamics and critical exponents of black holes with conformal anomaly, *Adv.High Energy Phys.* 2014, 739454(2014).
- [6] R. G. Cai, Thermodynamics of conformal anomaly corrected black holes in AdS space, *Phys. Lett. B* **733**, 183-189(2014).
- [7] D. Kastor, S. Ray, and J. Traschen, Enthalpy and the mechanics of AdS black holes, *Class. Quant. Grav.* **26**, 195011(2009).
- [8] B. Dolan, The cosmological constant and the black hole equation of state, *Class. Quant. Grav.* **28**, 125020(2011).
- [9] B. P. Dolan, Pressure and volume in the first law of black hole thermodynamics, *Class. Quant. Grav.* **28**, 235017(2011).
- [10] B. P. Dolan, Compressibility of rotating black holes, *Phys. Rev. D* **84**, 127503(2011).
- [11] M. Cvetič, G. Gibbons, D. Kubiznak, and C. Pope, Black hole enthalpy and an entropy inequality for the thermodynamic volume, *Phys. Rev. D* **84**, 024037(2011).
- [12] B. P. Dolan, Thermodynamic stability of asymptotically anti-de Sitter rotating black holes in higher dimensions, arXiv:1403.1507.
- [13] D. Kubizňák and R. B. Mann, *P-V* criticality of charged AdS black holes, *JHEP* **1207**, 033(2012).
- [14] S. Gunasekaran, R. B. Mann and D. Kubizňák, Extended phase space thermodynamics for charged and rotating black holes and Born-Infeld vacuum polarization, *JHEP* **1211**, 110(2012).
- [15] A. Belhaj, M. Chabab, H. El Moumni and M. B. Sedra, On thermodynamics of AdS black holes in arbitrary dimensions, *Chin. Phys. Lett.* **29**, 100401(2012) .
- [16] S. Chen, X. Liu, C. Liu and J. Jing, *P-V* criticality of AdS black hole in $f(R)$ gravity, *Chin. Phys. Lett.* **30**, 060401(2013).
- [17] S. H. Hendi and M. H. Vahidinia, Extended phase space thermodynamics and *P-V* criticality of black holes with nonlinear source, *Phys. Rev. D* **88**, 084045(2013).
- [18] E. Spallucci and A. Smailagic, Maxwell’s equal area law for charged Anti-deSitter black holes, *Phys. Lett. B* **723**, 436-441(2013).
- [19] R. Zhao, H. H. Zhao, M. S. Ma and L. C. Zhang, On the critical phenomena and thermodynamics of charged topological dilaton AdS black holes, *Eur. Phys. J. C* **73**, 2645(2013).
- [20] N. Altamirano, D. Kubizňák and R. Mann, Reentrant phase transitions in rotating AdS black holes, *Phys. Rev. D* **88**, 101502(2013).
- [21] R. G. Cai, L. M. Cao, L. Li and R. Q. Yang, *P-V* criticality in the extended phase space of Gauss-Bonnet black holes in AdS space, *JHEP* **1309**, 005(2013).

- [22] J. X. Mo and W. B. Liu, Ehrenfest scheme for P - V criticality in the extended phase space of black holes, Phys. Lett. B 727, 336-339 (2013).
- [23] J. X. Mo, X. X. Zeng, G. Q. Li, X. Jiang, W. B. Liu, A unified phase transition picture of the charged topological black hole in Hořava-Lifshitz gravity, JHEP 1310, 056(2013).
- [24] J. X. Mo and W. B. Liu, P - V Criticality of topological black holes in Lovelock-Born-Infeld gravity, Eur. Phys. J. C 74, 2836(2014).
- [25] J. X. Mo and W. B. Liu, Ehrenfest scheme for P - V criticality of higher dimensional charged black holes, rotating black holes and Gauss-Bonnet AdS black holes, Phys. Rev. D 89, 084057(2014).
- [26] N. Altamirano, D. Kubizňák, R. Mann and Z. Sherkatghanad, Kerr-AdS analogue of tricritical point and solid/liquid/gas phase transition, Class. Quant. Grav. 31(4), 042001(2014).
- [27] D. C. Zou, S. J. Zhang and B. Wang, Critical behavior of Born-Infeld AdS black holes in the extended phase space thermodynamics, Phys. Rev. D 89, 044002 (2014).
- [28] N. Altamirano, D. Kubizňák, R. Mann and Z. Sherkatghanad, Thermodynamics of rotating black holes and black rings: phase transitions and thermodynamic volume, Galaxies 2, 89-159(2014).
- [29] M. S. Ma, H. H. Zhao, L. C. Zhang and R. Zhao, Existence condition and phase transition of Reissner-Nordström-de Sitter black hole, Int. J. Mod. Phys. A 29, 1450050(2014).
- [30] S. W. Wei and Y. X. Liu, Triple points and phase diagrams in the extended phase space of charged Gauss-Bonnet black holes in AdS space, arXiv:1402.2837.
- [31] D. Kubizňák and R. B. Mann, Black hole chemistry, arXiv:1404.2126.
- [32] D. C. Zou, Y. Liu and B. Wang, Critical behavior of charged Gauss-Bonnet AdS black holes in the grand canonical ensemble, arXiv:1404.5194.
- [33] A. Belhaj, M. Chabab, H. EL Moumni, K. Masmar and M. B. Sedra, Ehrenfest Scheme of Higher Dimensional Topological AdS Black Holes in Lovelock-Born-Infeld Gravity, arXiv:1405.3306.
- [34] H. Xu, W. Xu and L. Zhao, Extended phase space thermodynamics for third order Lovelock black holes in diverse dimensions, arXiv:1405.4143.
- [35] W. Xu and L. Zhao, Critical phenomena of static charged AdS black holes in conformal gravity, arXiv:1405.7665.
- [36] C. V. Johnson, The Extended Thermodynamic Phase Structure of Taub-NUT and Taub-Bolt, arXiv:1406.4533.
- [37] A. M. Frassino, D. Kubizňák, R. B. Mann and F. Simovic, Multiple Reentrant Phase Transitions and Triple Points in Lovelock Thermodynamics, arXiv:1406.7015.
- [38] B. P. Dolan, A. Kostouki, D. Kubiznak and R. B. Mann, Isolated critical point from Lovelock gravity, arXiv:1407.4783.
- [39] B. P. Dolan, Black holes and Boyle's law-the thermodynamics of the cosmological constant, arXiv:1408.4023.